Hybrid Probabilistic Inference with Logical Constraints: Tractability and Message Passing

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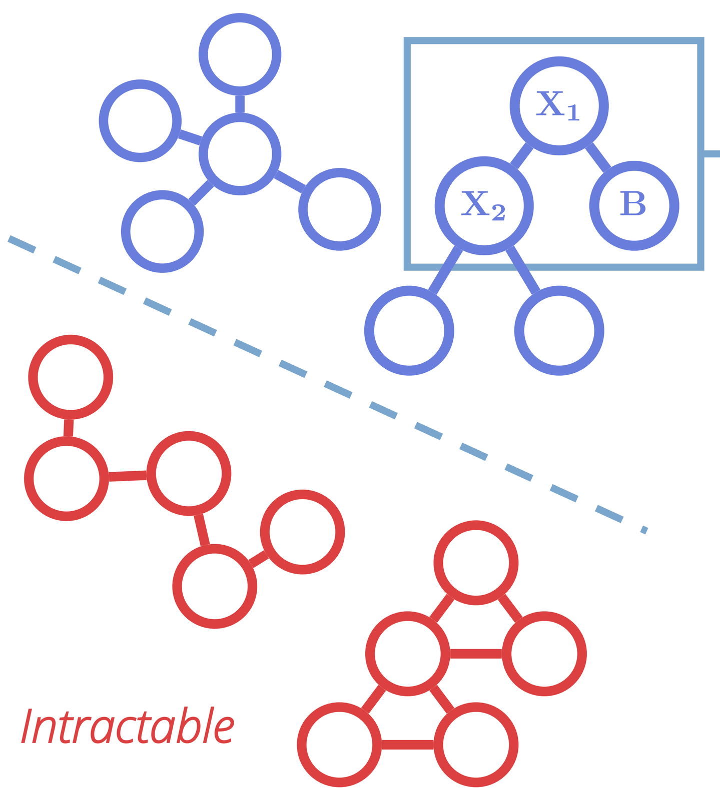
Hardness **: Treewidth, Diameter**



Logical Constraints **: SMT(*LRA*)**

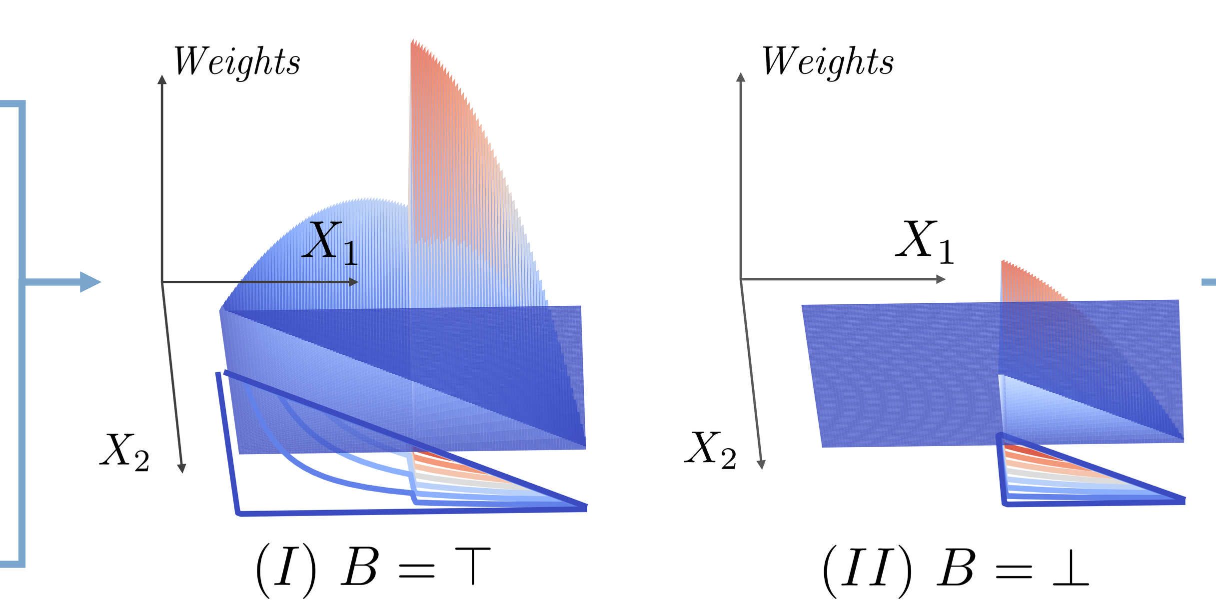
Formula

$0 \leq X_1 \leq 2$
$\wedge\ 0 \leq X_2 \leq 2$
$\wedge\ X_1 + X_2 \leq 2$ (ℓ_1)
$\wedge\ B \vee (1 \leq X_1)$ (ℓ_2, ℓ_3)

Per-Literal Weight

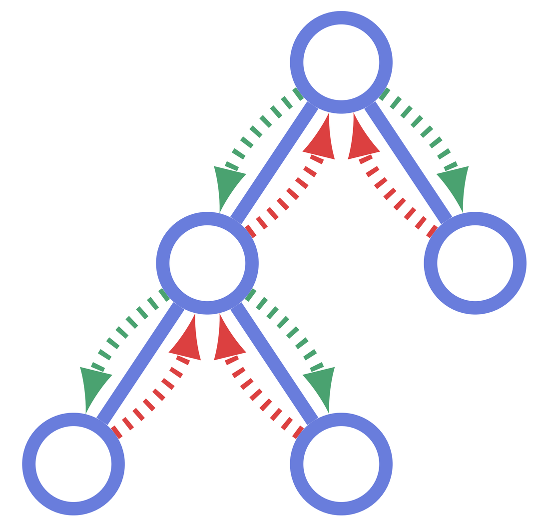
$w_{\ell_1}(x_1, x_2) = x_1 \cdot x_2$
$w_{\ell_2}(B) = 3$
$w_{\ell_3}(x_1) = 2$

Regions of weighted satisfying assignments



Probabilistic Inference **: WMI**

$$Pr(B) = \frac{\mathsf{WMI}(I)}{\mathsf{WMI}(I) + \mathsf{WMI}(II)}$$



Up-/Down-ward Messages in piecewise polynomial

WMI & MI

Weighted Model Integration (WMI) is a framework for hybrid probabilistic inference with logical constraints [1].

$$\mathsf{WMI}(\Delta, w; \mathbf{X}, \mathbf{B}) \triangleq \sum_{\boldsymbol{b} \in \mathbb{B}^{|\mathbf{B}|}} \int_{\Delta(\boldsymbol{x}, \boldsymbol{b})} w(\boldsymbol{x}, \boldsymbol{b})\, d\boldsymbol{x}$$

i.e. integration over the weighted assignments to hybrid variables, that satisfies a given logic constraint.

Logical constraints are Satisfiability Modulo Theories (SMT) formulas, as combinations of *Boolean* literals & *linear real arithmetic* (LRA) literals in Conjunctive Normal Form (CNF).

Primal graphs for SMT(LRA) formula: nodes ⇔ variables, edges ⇔ clauses.

Per-literal weights assign weight if the literal is SAT; otherwise assign one. Together they define a **joint weight**.

Answer queries

$$\Pr(q) = \frac{\mathsf{WMI}(\Delta \wedge q, w; \mathbf{X}, \mathbf{B})}{\mathsf{WMI}(\Delta, w; \mathbf{X}, \mathbf{B})}$$

WMI-to-MI reduction [2]

i. Hybrid WMI to real WMI by encoding Boolean literals into LRA literals.
ii. Real WMI to real MI by encoding polynomial per-literal weights into LRA constraints.

Contributions

- First we give *necessary* conditions for tractable MI problems by hardness proofs.
- Further we give sufficient conditions for tractability by proposing an efficient message passing scheme MP-MI for MI.

Hardness

What MI problems are intractable?

TLDR: Treewidth > 1 or Diameter O(n).

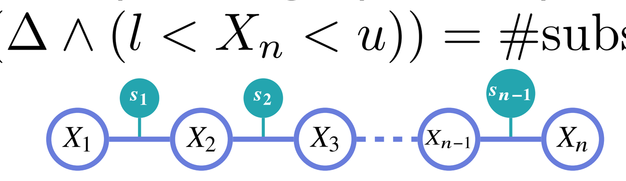
Thm. 1 Computing MI(Δ) of an SMT(LRA) formula Δ whose primal graph is a tree with diameter *O*(n) is #P-hard, with n = #variables.

Thm. 2 Computing MI(Δ) of an SMT(LRA) formula Δ whose primal graph has treewidth two and diameter *O*(log n) is #P-hard, with n = #variables.

Proof sketch: Both are proved by reducing Subset Sum Problem (SSP) to MI problems:

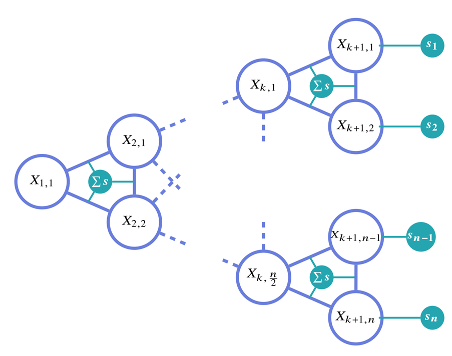
- For Thm.1, we construct an SMT(LRA) formula with primal graph as a path, s.t.

$$n^n \mathsf{MI}(\Delta \wedge (l < X_n < u)) = \#\text{subsets}$$



- For Thm. 2, we construct an SMT(LRA) formula with primal graph having treewidth two and diameter O(log n), s.t.

$$(2n)^{2n-1} \mathsf{MI}(\Delta \wedge (l < X_{1,1} < u)) = \#\text{subsets}$$



MP-MI

Exact message passing scheme for MI, with messages & beliefs in piecewise polynomials.

Messages

$$\mathsf{m}_{i \to j}(x_j) = \int_{\mathbb{R}} [\![x_i, x_j \models \Delta_{i,j}]\!][\![x_i \models \Delta_i]\!] \times \prod_{c \in \mathsf{neigh}(i) \setminus \{j\}} \mathsf{m}_{c \to i}(x_i)\, dx_i$$

Beliefs

$$\mathsf{b}_i^+(x_i) = \prod_{c \in \mathsf{ch}(i)} \mathsf{m}_{c \to i}(x_i), \quad \mathsf{b}_i(x_i) = \mathsf{b}_i^-(x_i) = \prod_{c \in \mathsf{neigh}(i)} \mathsf{m}_{c \to i}(x_i)$$

Algorithm 1 MP-MI(Δ) – Message Passing Model Integration

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1: V_up ← sort nodes in G_Δ, children before parents
2: for each X_i ∈ V_up do send-message(X_i, X_parent(X_i)) end      ▹ upward pass
3: V_down ← sort nodes in G_Δ, parents before children
4: for each X_i ∈ V_down do                                            ▹ downward pass
5:     for each X_c ∈ ch(X_i) do send-message(X_i, X_c) end
6: Return {b_i}_{i:X_i ∈ G_Δ}
send-message(X_i, X_j)
1: b_i ← compute-beliefs
2: P ← critical-points(b_i, Δ_i, Δ_{i,j}),     I ← intervals-from-points(P)
3: for interval [l,u] ∈ I consistent with formula Δ_i ∧ Δ_{i,j} do
4:     ⟨l_s, u_s, f⟩ ← symbolic-bounds(b_i, [l,u], Δ_{i,j})
5:     f' ← ∫_{l_s}^{u_s} f(x_i) dx_i,     m_{i→j} ← m_{i→j} ∪ ⟨l,u,f'⟩
6: Return m_{i→j}
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When the message passing scheme is done, the MI can be computed by

$$\mathsf{MI}(\Delta) = \int_{\mathbb{R}} [\![x_i \models \Delta_i]\!] \cdot \mathsf{b}_i(x_i)\, dx_i.$$

And also *amortize* queries:

Node queries $\mathsf{MI}(\Delta \wedge q) = \int_{\mathbb{R}} [\![x_i \models q]\!][\![x_i \models \Delta_i]\!] \mathsf{b}_i(x_i) dx_i.$

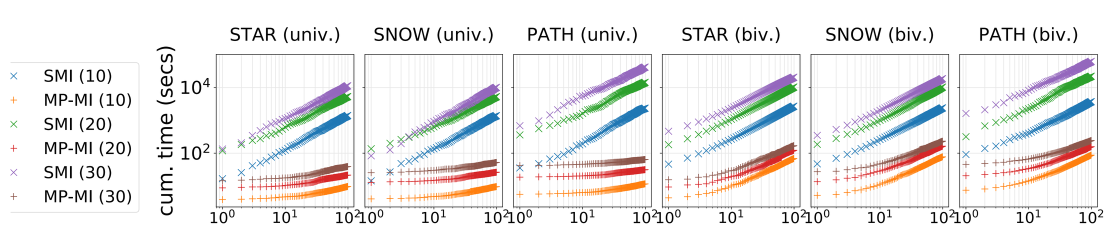
Edge queries

$$\mathsf{m}^*_{j \to i}(x_i) = \int_{\mathbb{R}} \mathsf{b}_j(x_j)/\mathsf{m}_{i \to j}(x_j) \times [\![x_i, x_j \models \Delta_{i,j} \wedge q]\!][\![x_j \models \Delta_j]\!]\, dx_j$$

Moment queries $\mathbb{E}[X_i^k] = \frac{1}{\mathsf{MI}(\Delta)} \int_{\mathbb{R}} [\![x_i \models \Delta_i]\!] \times x_i^k \mathsf{b}_i(x_i)\, dx_i$

Experiments

Time for answer multiple queries to SMT(LRA) formulas with tree primal graphs of diameters *O*(1), *O*(log n) & *O*(n) respectively.



<- **Paper available here**

KR2ML @ NeurIPS 19,
Vancouver, Canada

References: [1] Vaishak Belle, Andrea Passerini, and Guy Van den Broeck. Probabilistic inference in hybrid domains by weighted model integration. IJCAI, 2015.
[2] Zhe Zeng and Guy Van den Broeck. Efficient search-based weighted model integration. UAI, 2019.

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